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Definition Let (X, d_1) and (Y, d_2) be metric spaces. Let $S \subseteq X$. A function $f: S \to Y$ is **uniformly continuous** on S if $\forall\, \varepsilon > 0$ and $\forall\, x \in S$, $\exists\, \delta > 0$ such that

$$d_2(f(x), f(y)) < \varepsilon \text{ when } d_1(x, y) < \delta$$

$(x, y \in S)$

If f is merely continuous,

δ depends on x and ε.

If f is uniformly continuous,

δ depends only on ε.

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Theorem: If $S \subseteq (X, d_1)$ is compact and $f: S \to (Y, d_2)$ is continuous, then f is uniformly continuous on S.

proof: Let $\varepsilon > 0$. Then $\forall\, x \in S$, $\exists\, \delta_x > 0$ such that if $y \in S$ and

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if $d_1(x,y) < \delta_x$, then

$d_2(f(x), f(y)) < \varepsilon/2$.

Consider

$$S \subseteq \bigcup_{x \in S} B(x, \delta_x/2).$$

Then S is compact, so

$\exists\ x_1, x_2, \ldots, x_n$ with

$$S \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\delta_{x_i}}{2}\right)$$

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Let $\delta = \frac{1}{2}\min\left\{\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\right\}$.

I claim this works.

Take $x, y \in S$, suppose

$d_1(x,y) < \delta$.

We know $\exists\ i$, $1 \le i \le n$,

$x \in B\left(x_i, \frac{\delta_{x_i}}{2}\right)$.

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$$d_1(y, x_i)$$

$$\leq d_1(y,x) + d_1(x, x_i)$$

$$< \delta + \frac{\delta x_i}{2}$$

$$< \frac{\delta x_i}{4} + \frac{\delta x_i}{2} < \delta_{x_i}.$$

Then $d_1(x, x_i), d_1(y, x_i) < \delta_{x_i}$,

so we are done by

the triangle inequality.

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$$d_2(f(x), f(y))$$

$$\leq d_2(f(x), f(x_i))$$

$$+ d_2(f(x_i), f(y))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

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In english:

"Continuous functions on a compact set are uniformly continuous."

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Recall: Intermediate value theorem, differentiability on a closed interval.

IVT: If $f : [a,b] \to \mathbb{R}$ is continuous (usual metric), then f assumes every value between $f(a)$ and $f(b)$ somewhere on $[a,b]$.

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Question: Is the converse true?

No!

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Recall f is differentiable on $[a,b]$ if $\exists$ an open set $O \supseteq [a,b]$ such that f is differentiable on O.

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Theorem: (derivatives + IVT)

Let $f: [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$.

Then f' satisfies the IVT.

proof: Wednesday.

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Example:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider f on the interval $[-1, 1]$. For $x \neq 0$,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

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$$\lim_{x \to 0} f'(x)$$

$$= \lim_{x \to 0} \left(2x\sin(1/x) - \cos(1/x) \right)$$

$$2x\sin(1/x) \to 0 \qquad \cos(1/x): \text{limit does not exist}$$

Therefore, $\lim_{x \to 0} f'(x)$ does not exist, so f' is not continuous at $x = 0$.

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But

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h}$$

$$= \lim_{h \to 0} h \sin(1/h)$$

$$= 0$$

So f is differentiable everywhere, yet f' is discontinuous at $x = 0$.

Reading for Wednesday:

Sections 7.1 and 7.2